CHAPTER 5

# Method of Weighted Residuals

## 5.1 INTRODUCTION

Chapters 2, 3, and 4 introduced some of the basic concepts of the finite element method in terms of the so-called line elements. The linear elastic spring, the bar element and the flexure element are line elements because structural properties can be described in terms of a single spatial variable that identifies position along the longitudinal axis of the element. The displacement-force relations for the line elements are straightforward, as these relations are readily described using only the concepts of elementary strength of materials. To extend the method of finite element analysis to more general situations, particularly nonstructural applications, additional mathematical techniques are required. In this chapter, the method of weighted residuals is described in general and Galerkin's method of weighted residuals [1] is emphasized as a tool for finite element formulation for essentially any field problem governed by a differential equation.

## 5.2 METHOD OF WEIGHTED RESIDUALS

It is a basic fact that most practical problems in engineering are governed by differential equations. Owing to complexities of geometry and loading, rarely are exact solutions to the governing equations possible. Therefore, approximate techniques for solving differential equations are indispensable in engineering analysis. Indeed, the finite element method is such a technique. However, the finite element method is based on several other, more-fundamental, approximate techniques, one of which is discussed in detail in this section and subsequently applied to finite element formulation.

The *method of weighted residuals* (MWR) is an approximate technique for solving boundary value problems that utilizes *trial functions* satisfying the prescribed boundary conditions and an integral formulation to minimize error, in an average sense, over the problem domain. The general concept is described here in terms of the one-dimensional case but, as is shown in later chapters, extension to two and three dimensions is relatively straightforward. Given a differential equation of the general form

$$D[y(x), x] = 0 \qquad a < x < b \tag{5.1}$$

subject to homogeneous boundary conditions

$$y(a) = y(b) = 0 \tag{5.2}$$

the method of weighted residuals seeks an approximate solution in the form

$$y^*(x) = \sum_{i=1}^{n} c_i N_i(x) \tag{5.3}$$

where $y^*$ is the approximate solution expressed as the product of $c_i$ unknown, constant parameters to be determined and $N_i(x)$ trial functions. The major requirement placed on the trial functions is that they be *admissible functions;* that is, the trial functions are continuous over the domain of interest and satisfy the specified boundary conditions exactly. In addition, the trial functions should be selected to satisfy the "physics" of the problem in a general sense. Given these somewhat lax conditions, it is highly unlikely that the solution represented by Equation 5.3 is exact. Instead, on substitution of the assumed solution into the differential Equation 5.1, a residual error (hereafter simply called *residual*) results such that

$$R(x) = D[y^*(x), x] \neq 0 \tag{5.4}$$

where $R(x)$ is the residual. Note that the residual is also a function of the unknown parameters $c_i$. The method of weighted residuals requires that the unknown parameters $c_i$ be evaluated such that

$$\int_a^b w_i(x) R(x)\,\mathrm{d}x = 0 \qquad i = 1, n \tag{5.5}$$

where $w_i(x)$ represents $n$ arbitrary weighting functions. We observe that, on integration, Equation 5.5 results in $n$ algebraic equations, which can be solved for the $n$ values of $c_i$. Equation 5.5 expresses that the sum (integral) of the weighted residual error over the domain of the problem is zero. Owing to the requirements placed on the trial functions, the solution is exact at the end points (the boundary conditions must be satisfied) but, in general, at any interior point the residual error is nonzero. As is subsequently discussed, the MWR may capture the exact solution under certain conditions, but this occurrence is the exception rather than the rule.

Several variations of MWR exist and the techniques vary primarily in how the weighting factors are determined or selected. The most common techniques are point collocation, subdomain collocation, least squares, and Galerkin's method [1]. As it is quite simple to use and readily adaptable to the finite element method, we discuss only Galerkin's method.

In Galerkin's weighted residual method, the weighting functions are chosen to be identical to the trial functions; that is,

$$w_i(x) = N_i(x) \qquad i = 1, n \tag{5.6}$$

Therefore, the unknown parameters are determined via

$$\int_a^b w_i(x)R(x)\,\mathrm{d}x = \int_a^b N_i(x)R(x) = 0 \qquad i = 1, n \tag{5.7}$$

again resulting in $n$ algebraic equations for evaluation of the unknown parameters. The following examples illustrate details of the procedure.

**EXAMPLE 5.1**

Use Galerkin's method of weighted residuals to obtain an approximate solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 10x^2 = 5 \qquad 0 \le x \le 1$$

with boundary conditions $y(0) = y(1) = 0$.

■ **Solution**

The presence of the quadratic term in the differential equation suggests that trial functions in polynomial form are suitable. For homogeneous boundary conditions at $x = a$ and $x = b$, the general form

$$N(x) = (x - x_a)^p (x - x_b)^q$$

with $p$ and $q$ being positive integers greater than zero, automatically satisfies the boundary conditions and is continuous in $x_a \le x \le x_b$. Using a single trial function, the simplest such form that satisfies the stated boundary conditions is

$$N_1(x) = x(x - 1)$$

Using this trial function, the approximate solution per Equation 5.3 is

$$y^*(x) = c_1 x(x - 1)$$

and the first and second derivatives are

$$\frac{\mathrm{d}y^*}{\mathrm{d}x} = c_1(2x - 1)$$

$$\frac{\mathrm{d}^2 y^*}{\mathrm{d}x^2} = 2c_1$$

respectively. (We see, at this point, that the selected trial solution does not satisfy the physics of the problem, since we have obtained a constant second derivative. The differential equation is such that the second derivative must be a quadratic function of $x$. Nevertheless, we continue the example to illustrate the procedure.)

Substitution of the second derivative of $y^*(x)$ into the differential equation yields the residual as

$$R(x; c_1) = 2c_1 - 10x^2 - 5$$

which is clearly nonzero. Substitution into Equation 5.7 gives

$$\int_0^1 x(x-1)(2c_1 - 10x^2 - 5)\,\mathrm{d}x = 0$$

which after integration yields $c_1 = 4$, so the approximate solution is obtained as

$$y^*(x) = 4x(x-1)$$

For this relatively simple example, we can compare the approximate solution result with the exact solution, obtained by integrating the differential equation twice as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \int \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\,\mathrm{d}x = \int (10x^2 + 5)\,\mathrm{d}x = \frac{10x^3}{3} + 5x + C_1$$

$$y(x) = \int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \left(\frac{10x^3}{3} + 5x + C_1\right)\mathrm{d}x = \frac{5x^4}{6} + \frac{5x^2}{2} + C_1 x + C_2$$

Applying the boundary condition $y(0) = 0$ gives $C_2 = 0$, while the condition $y(1) = 0$ becomes

$$\frac{5}{6} + \frac{5}{2} + C_1 = 0$$

from which $C_1 = -10/3$. Hence, the exact solution is given by

$$y(x) = \frac{5}{6}x^4 + \frac{5}{2}x^2 - \frac{10}{3}x$$

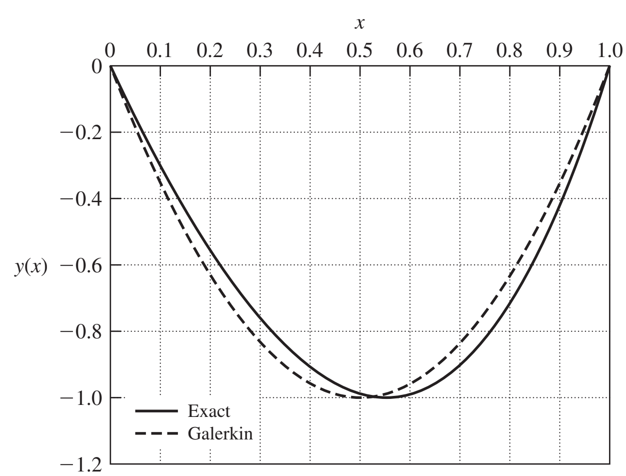


**Figure 5.1** Solutions to Example 5.1.

A graphical comparison of the two solutions is depicted in Figure 5.1, which shows that the approximate solution is in reasonable agreement with the exact solution. However, note that the one-term approximate solution is symmetric over the interval of interest. That this is not correct can be seen by examining the differential equation. The prime driving "force" is the quadratic term in $x$; therefore, it is unlikely that the solution is symmetric. The following example expands the solution and shows how the method approaches the exact solution.

**EXAMPLE 5.2**

Obtain a two-term Galerkin solution for the problem of Example 5.1 using the trial functions

$$N_1(x) = x(x-1) \qquad N_2(x) = x^2(x-1)$$

■ **Solution**

The two-term approximate solution is

$$y^* = c_1 x(x-1) + c_2 x^2 (x-1)$$

and the second derivative is

$$\frac{d^2 y^*}{dx^2} = 2c_1 + 2c_2(3x-1)$$

Substituting into the differential equation, we obtain the residual

$$R(x; c_1, c_2) = 2c_1 + 2c_2(3x-1) - 10x^2 - 5$$

Using the trial functions as the weighting functions per Galerkin's method, the residual equations become

$$\int_0^1 x(x-1)[2c_1 + 2c_2(3x-1) - 10x^2 - 5]\, dx = 0$$

$$\int_0^1 x^2(x-1)[2c_1 + 2c_2(3x-1) - 10x^2 - 5]\, dx = 0$$

After integration and simplification, we obtain the algebraic equations

$$-\frac{c_1}{3} - \frac{c_2}{6} + \frac{4}{3} = 0$$

$$-\frac{c_1}{6} - \frac{2c_2}{15} + \frac{3}{4} = 0$$

Simultaneous solution results in

$$c_1 = \frac{19}{6} \qquad c_2 = \frac{5}{3}$$

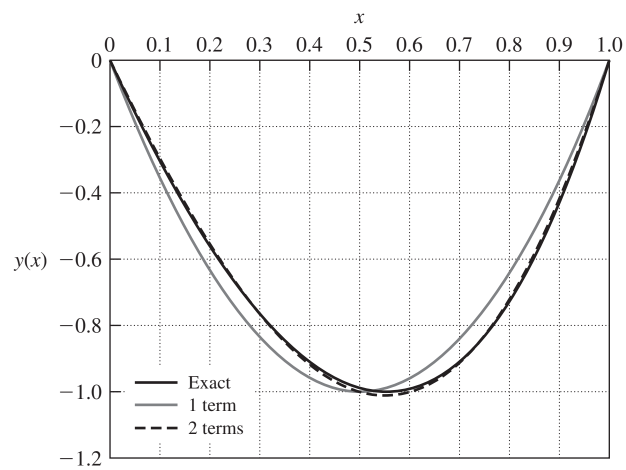


**Figure 5.2** Solutions to Example 5.2.

so the two-term approximate solution is

$$y^* = \frac{19}{6}x(x-1) + \frac{5}{3}x^2(x-1) = \frac{5}{3}x^3 + \frac{3}{2}x^2 - \frac{19}{6}x$$

For comparison, the exact, one-term and two-term solutions are plotted in Figure 5.2. The differences in the exact and two-term solutions are barely discernible.

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**EXAMPLE 5.3**

Use Galerkin's method of weighted residuals to obtain a one-term approximation to the solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 4x \qquad 0 \le x \le 1$$

with boundary conditions $y(0) = 0$, $y(1) = 1$.

■ **Solution**

Here the boundary conditions are not homogeneous, so a modification is required. Unlike the case of homogeneous boundary conditions, it is not possible to construct a trial solution of the form $c_1 N_1(x)$ that satisfies both stated boundary conditions. Instead, we assume a trial solution as

$$y^* = c_1 N_1(x) + f(x)$$

where $N_1(x)$ satisfies the homogeneous boundary conditions and $f(x)$ is chosen to satisfy the nonhomogeneous condition. (Note that, if both boundary conditions were nonhomogeneous, two such functions would be included.) One such solution is

$$y^* = c_1 x(x-1) + x$$

which satisfies $y(0) = 0$ and $y(1) = 1$ identically.

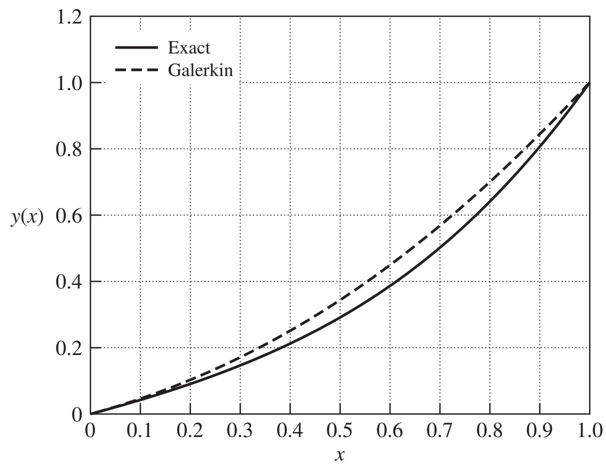


**Figure 5.3** Solutions to Example 5.3.

Substitution into the differential equation results in the residual

$$R(x; c_1) = \frac{\mathrm{d}^2 y^*}{\mathrm{d}x^2} + y^* - 4x = 2c_1 + c_1 x^2 - c_1 x + x - 4x = c_1 x^2 - c_1 x + 2c_1 - 3x$$

and the weighted residual integral becomes

$$\int_0^1 N_1(x) R(x; c_1)\,\mathrm{d}x = \int_0^1 x(x-1)(c_1 x^2 + c_1 x - 2c_1 - 3x)\,\mathrm{d}x = 0$$

While algebraically tedious, the integration is straightforward and yields

$$c_1 = 5/6$$

so the approximate solution is

$$y^*(x) = \frac{5}{6}x(x-1) + x = \frac{5}{6}x^2 + \frac{1}{6}x$$

As in the previous example, we have the luxury of comparing the approximate solution to the exact solution, which is

$$y(x) = 4x - 3.565 \sin x$$

The approximate solution and the exact solution are shown in Figure 5.3 for comparison. Again, the agreement is observed to be reasonable but could be improved by adding a second trial function.

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How does one know when the MWR solution is accurate enough? That is, how do we determine whether the solution is close to the exact solution? This question of *convergence* must be addressed in all approximate solution techniques. If we do not know the exact solution, and we seldom do, we must develop some criterion to determine accuracy. In general, for the method of weighted residuals, the procedure is to continue obtaining solutions while increasing the number of trial functions and note the behavior of the solution. If the solution changes very little as we increase the number of trial functions, we can say that the solution converges. Whether the solution converges to the correct solution is yet another question. While beyond the scope of this book, a large body of theoretical mathematics addresses the questions of convergence and whether the convergence is to the correct solution. In the context of this work, we assume that a converging solution converges to the correct solution. Certain checks, external to the solution procedure, can be made to determine the "reasonableness" of a numerical solution in the case of physical problems. These checks include equilibrium, energy balance, heat and fluid flow balance, and others discussed in following chapters.

In the previous examples, we used trial functions "concocted" to satisfy boundary conditions automatically but not based on a systematic procedure. While absolutely nothing is wrong with this approach, we now present a procedure, based on polynomial trial functions, that gives a method for increasing the number of trial functions systematically and, hence, aids in examining convergence. The procedure is illustrated in the context of the following example.

## EXAMPLE 5.4

Solve the problem of Examples 5.1 and 5.2 by assuming a general polynomial form for the solution as

$$y^*(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

### ■ Solution

For a first trial, we take only the quadratic form

$$y^*(x) = c_0 + c_1 x + c_2 x^2$$

and apply the boundary conditions to obtain

$$y^*(0) = 0 = c_0$$
$$y^*(1) = 0 = c_1 + c_2$$

The second boundary condition equations show that $c_1$ and $c_2$ are not independent if the homogeneous boundary condition is to be satisfied exactly. Instead, we obtain the *constraint* relation $c_2 = -c_1$. The trial solution becomes

$$y^*(x) = c_1 x + c_2 x^2 = c_1 x - c_1 x^2 = c_1 x(1 - x)$$

and is the same as the solution obtained in Example 5.1.

Next we add the cubic term and write the trial solution as

$$y^*(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

Application of the boundary conditions results in

$$y^*(0) = 0 = c_0$$
$$y^*(1) = 0 = c_1 + c_2 + c_3$$

so we have the constraint relation

$$c_1 + c_2 + c_3 = 0$$

Expressing the constraint as $c_3 = -(c_1 + c_2)$, the trial solution becomes

$$y^*(x) = c_1 x + c_2 x^2 + c_3 x^3 = c_1 x + c_2 x^2 - (c_1 + c_2)x^3 = c_1 x(1 - x^2) + c_2 x^2(1 - x)$$

and we have obtained two trial functions, each identically satisfying the boundary conditions. Determination of the constants for the two-term solution is left as and end-of-chapter exercise. Instead, we add the quartic term and examine the trial solution

$$y^*(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

and the boundary conditions give

$$c_0 = 0$$
$$c_1 + c_2 + c_3 + c_4 = 0$$

We use the constraint relation to eliminate (arbitrarily) $c_4$ to obtain

$$\begin{aligned} y^*(x) &= c_1 x + c_2 x^2 + c_3 x^3 - (c_1 + c_2 + c_3)x^4 \\ &= c_1 x(1 - x^3) + c_2 x^2(1 - x^2) + c_3 x^3(1 - x) \end{aligned}$$

Substituting into the differential equation, the residual is found to be

$$R(x; c_1, c_2, c_3) = -12c_1 x^2 + c_2(2 - 12x^2) + c_3(6x - 12x^2) - 10x^2 - 5$$

If we set the residual expression equal to zero and equate coefficients of powers of $x$, we find that the residual is exactly zero if

$$c_1 = -\frac{10}{3}$$
$$c_2 = \frac{5}{2}$$
$$c_3 = 0$$
$$c_4 = \frac{5}{6}$$

so that $y^*(x) = \frac{5}{6}x^4 + \frac{5}{2}x^2 - \frac{10}{3}x$ and we have obtained the exact solution.

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The procedure detailed in the previous example represents a systematic procedure for developing polynomial trial functions and is also applicable to the case of nonhomogeneous boundary conditions. Algebraically, the process is straightforward but becomes quite tedious as the number of trial functions is increased (i.e., the order of the polynomial). Having outlined the general technique of Galerkin's method of weighted residuals, we now develop Galerkin's finite element method based on MWR.

## 5.3 THE GALERKIN FINITE ELEMENT METHOD

The classic method of weighted residuals described in the previous section utilizes trial functions that are global; that is, each trial function must apply over the entire domain of interest and identically satisfy the boundary conditions. Particularly in the more practical cases of two- and three-dimensional problems governed by partial differential equations, "discovery" of appropriate trial functions and determination of the accuracy of the resulting solutions are formidable tasks. However, the concept of minimizing the residual error is readily adapted to the finite element context using the Galerkin approach as follows. For illustrative purposes, we consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + f(x) = 0 \qquad a \leq x \leq b \tag{5.8}$$

subject to boundary conditions

$$y(a) = y_a \qquad y(b) = y_b \tag{5.9}$$

The problem domain is divided into $M$ "elements" (Figure 5.4a) bounded by $M + 1$ values $x_i$ of the independent variable, so that $x_1 = x_a$ and $x_{M+1} = x_b$ to

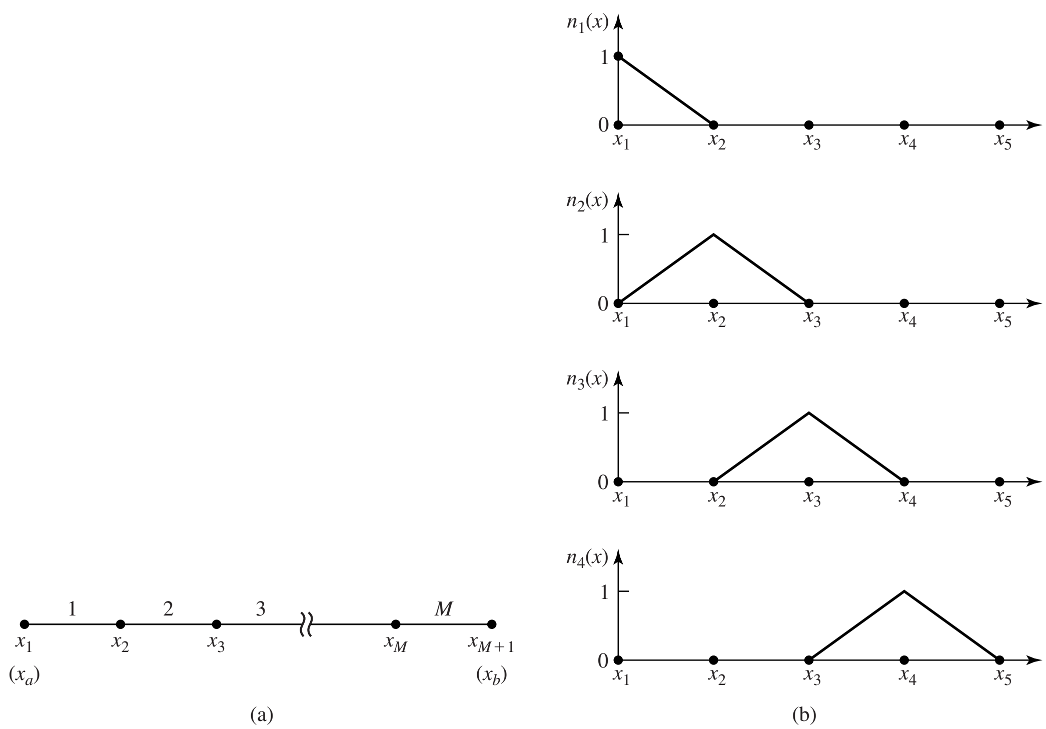



**Figure 5.4**
(a) Domain $x_a \leq x \leq x_b$ discretized into $M$ elements. (b) First four trial functions. Note the overlap of only two trial functions in each element domain.

ensure inclusion of the global boundaries. An approximate solution is assumed in the form

$$y^*(x) = \sum_{i=1}^{M+1} y_i n_i(x) \tag{5.10}$$

where $y_i$ is the value of the solution function at $x = x_i$ and $n_i(x)$ is a corresponding trial function. Note that, in this approach, the unknown constant parameters $c_i$ of the method of weighted residuals become unknown discrete values of the solution function evaluated at specific points in the domain. There also exists a major difference in the trial functions. As used in Equation 5.10, the trial functions $n_i(x)$ are nonzero over only a small portion of the global problem domain. Specifically, a trial function $n_i(x)$ is nonzero only in the interval $x_{i-1} < x < x_{i+1}$, and for ease of illustration, we use linear functions defined as follows:

$$\begin{aligned}
n_i(x) &= \frac{x - x_{i-1}}{x_i - x_{i-1}} \qquad x_{i-1} \le x \le x_i \\
n_i(x) &= \frac{x_{i+1} - x}{x_{i+1} - x_i} \qquad x_i \le x \le x_{i+1} \\
n_i(x) &= 0 \qquad x < x_{i-1} \qquad x > x_{i+1}
\end{aligned} \tag{5.11}$$

Clearly, in this case, the trial functions are simply linear interpolation functions such that the value of the solution $y(x)$ in $x_i < x < x_{i+1}$ is a linear combination of adjacent "nodal" values $y_i$ and $y_{i+1}$. The first four trial functions are as shown in Figure 5.4b, and we observe that, in the interval $x_2 \le x \le x_3$, for example, the approximate solution as given by Equation 5.10 is

$$y^*(x) = y_2 n_2(x) + y_3 n_3(x) = y_2 \frac{x_3 - x}{x_3 - x_2} + y_3 \frac{x - x_2}{x_3 - x_2} \tag{5.12}$$

(The trial functions used here are linear but higher-order functions can also be used, as is subsequently demonstrated by application of the technique to a beam element.)

Substitution of the assumed solution (5.10) into the governing Equation 5.8 yields the residual

$$R(x; y_i) = \sum_{i=1}^{M+1} \left[ \frac{\mathrm{d}^2 y^*}{\mathrm{d}x^2} + f(x) \right] = \sum_{i=1}^{M+1} \left[ \frac{\mathrm{d}^2}{\mathrm{d}x^2} \{ y_i n_i(x) \} + f(x) \right] \tag{5.13}$$

to which we apply Galerkin's weighted residual method, using each trial function as a weighting function, to obtain

$$\int_{x_a}^{x_b} n_j(x) R(x; y_i)\, \mathrm{d}x = \int_{x_a}^{x_b} n_j(x) \sum_{i=1}^{M+1} \left[ \frac{\mathrm{d}^2}{\mathrm{d}x^2} \{ y_i n_i(x) \} + f(x) \right] \mathrm{d}x = 0$$

$$j = 1, M + 1 \tag{5.14}$$

In light of Equation 5.11 and Figure 5.4b, we observe that, in any interval $x_j \le x \le x_{j+1}$, only two of the trial functions are nonzero. Taking this observation into account, Equation 5.14 can be expressed as

$$\int_{x_j}^{x_{j+1}} n_j(x)\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}(y_j n_j(x) + y_{j+1} n_{j+1}(x)) + f(x)\right]\mathrm{d}x = 0 \qquad j = 1, M+1 \tag{5.15}$$

Integration of Equation 5.15 yields $M + 1$ algebraic equations in the $M + 1$ unknown nodal solution values $y_j$, and these equations can be written in the matrix form

$$[K]\{y\} = \{F\} \tag{5.16}$$

where $[K]$ is the system "stiffness" matrix, $\{y\}$ is the vector of nodal "displacements" and $\{F\}$ is the vector of nodal "forces." Equation 5.14 is the formal statement of the Galerkin finite element method and includes both element formation and system assembly steps. Written in terms of integration over the full problem domain, this formulation clearly shows the mathematical basis in the method of weighted residuals. However, Equations 5.15 show that integration over only each element is required for each of the equations. We now proceed to examine separate element formulation based on Galerkin's method.

### 5.3.1 Element Formulation

If the exact solution for Equation 5.8 is obtained, then that solution satisfies the equation in any subdomain in $(a, b)$ as well. Consider the problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + f(x) = 0 \qquad x_j \le x \le x_{j+1} \tag{5.17}$$

where $x_j$ and $x_{j+1}$ are contained in $(a, b)$ and define the nodes of a finite element. The appropriate boundary conditions applicable to Equation 5.17 are

$$y(x_j) = y_j \qquad y(x_{j+1}) = y_{j+1} \tag{5.18}$$

and these are the unknown values of the solution at the end points of the subdomain. Next we propose an approximate solution of the form

$$y^{(e)}(x) = y_j N_1(x) + y_{j+1} N_2(x) \tag{5.19}$$

where superscript $(e)$ indicates that the solution is for the finite element and the interpolation functions are now defined as

$$N_1(x) = \frac{x_{j+1} - x}{x_{j+1} - x_j} \qquad x_j \le x \le x_{j+1} \tag{5.20a}$$

$$N_2(x) = \frac{x - x_j}{x_{j+1} - x_j} \qquad x_j \le x \le x_{j+1} \tag{5.20b}$$

Note the relation between the interpolation functions defined in Equation 5.20 and the trial functions in Equation 5.11. The interpolation functions correspond to the overlapping portions of the trial functions applicable in a single element domain. Also note that the interpolation functions satisfy the conditions

$$\begin{aligned} N_1(x = x_j) &= 1 \qquad N_1(x = x_{j+1}) = 0 \\ N_2(x = x_j) &= 0 \qquad N_2(x = x_{j+1}) = 1 \end{aligned} \tag{5.21}$$

such that the element boundary (nodal) conditions, Equation 5.18, are identically satisfied. Substitution of the assumed solution into Equation 5.19 gives the residual as

$$R^{(e)}(x; y_j, y_{j+1}) = \frac{\mathrm{d}^2 y^{(e)}}{\mathrm{d}x^2} + f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}[y_j N_1(x) + y_{j+1} N_2(x)] + f(x) \neq 0 \tag{5.22}$$

where the superscript is again used to indicate that the residual is for the *element*. Applying the Galerkin weighted residual criterion results in

$$\int_{x_j}^{x_{j+1}} N_i(x) R^{(e)}(x; y_j, y_{j+1})\,\mathrm{d}x = \int_{x_j}^{x_{j+1}} N_i(x)\left[\frac{\mathrm{d}^2 y^{(e)}}{\mathrm{d}x^2} + f(x)\right]\mathrm{d}x = 0 \qquad i = 1, 2 \tag{5.23}$$

or

$$\int_{x_j}^{x_{j+1}} N_i(x)\frac{\mathrm{d}^2 y^{(e)}}{\mathrm{d}x^2}\,\mathrm{d}x + \int_{x_j}^{x_{j+1}} N_i(x) f(x)\,\mathrm{d}x = 0 \qquad i = 1, 2 \tag{5.24}$$

as the element residual equations.

Applying integration by parts to the first integral results in

$$N_i(x)\frac{\mathrm{d}y^{(e)}}{\mathrm{d}x}\bigg|_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}y^{(e)}}{\mathrm{d}x}\,\mathrm{d}x + \int_{x_j}^{x_{j+1}} N_i(x) f(x)\,\mathrm{d}x = 0 \qquad i = 1, 2 \tag{5.25}$$

which, after evaluation of the nonintegral term and rearranging is equivalent to the two equations, is

$$\int_{x_j}^{x_{j+1}} \frac{\mathrm{d}N_1}{\mathrm{d}x}\frac{\mathrm{d}y^{(e)}}{\mathrm{d}x}\,\mathrm{d}x = \int_{x_j}^{x_{j+1}} N_1(x) f(x)\,\mathrm{d}x + \frac{\mathrm{d}y^{(e)}}{\mathrm{d}x}\bigg|_{x_j} \tag{5.26a}$$

$$\int_{x_j}^{x_{j+1}} \frac{\mathrm{d}N_2}{\mathrm{d}x}\frac{\mathrm{d}y^{(e)}}{\mathrm{d}x}\,\mathrm{d}x = \int_{x_j}^{x_{j+1}} N_2(x) f(x)\,\mathrm{d}x - \frac{\mathrm{d}y^{(e)}}{\mathrm{d}x}\bigg|_{x_{j+1}} \tag{5.26b}$$

Note that, in arriving at the form of Equation 5.26, explicit use has been made of Equation 5.21 in evaluation of the interpolation functions at the element nodes.

Integration of Equation 5.24 by parts results in three benefits [2]:

1. The highest order of the derivatives appearing in the element equations has been reduced by one.
2. As will be observed explicitly, the stiffness matrix was made symmetric. If we did not integrate by parts, one of the trial functions in each equation would be differentiated twice and the other trial function not differentiated at all.
3. Integration by parts introduces the gradient boundary conditions at the element nodes. The physical significance of the gradient boundary conditions becomes apparent in subsequent physical applications.

Setting $j = 1$ for notational simplicity and substituting Equation 5.19 into Equation 5.26 yields

$$\int_{x_1}^{x_2} \frac{\mathrm{d}N_1}{\mathrm{d}x}\left[y_1\frac{\mathrm{d}N_1}{\mathrm{d}x} + y_2\frac{\mathrm{d}N_2}{\mathrm{d}x}\right]\mathrm{d}x = \int_{x_1}^{x_2} N_1(x)f(x)\,\mathrm{d}x + \left.\frac{\mathrm{d}y^{(e)}}{\mathrm{d}x}\right|_{x_1} \tag{5.27a}$$

$$\int_{x_1}^{x_2} \frac{\mathrm{d}N_2}{\mathrm{d}x}\left[y_1\frac{\mathrm{d}N_1}{\mathrm{d}x} + y_2\frac{\mathrm{d}N_2}{\mathrm{d}x^2}\right]\mathrm{d}x = \int_{x_1}^{x_2} N_2(x)f(x)\,\mathrm{d}x - \left.\frac{\mathrm{d}y^{(e)}}{\mathrm{d}x}\right|_{x_2} \tag{5.27b}$$

which are of the form

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{5.28}$$

The terms of the coefficient (element stiffness) matrix are defined by

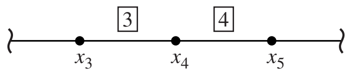
$$k_{ij} = \int_{x_1}^{x_2} \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}N_j}{\mathrm{d}x}\,\mathrm{d}x \qquad i, j = 1, 2 \tag{5.29}$$

and the element nodal forces are given by the right-hand sides of Equation 5.27.

If the described Galerkin procedure for element formulation is followed and the system equations are assembled in the usual manner of the direct stiffness method, the resulting system equations are identical in every respect to those obtained by the procedure represented by Equation 5.13. It is important to observe that, during the assembly process, when two elements are joined at a common node as in Figure 5.5, for example, the assembled system equation for the node contains a term on the right-hand side of the form

$$-\left.\frac{\mathrm{d}y^{(3)}}{\mathrm{d}x}\right|_{x_4} + \left.\frac{\mathrm{d}y^{(4)}}{\mathrm{d}x}\right|_{x_4} \tag{5.30}$$

*If* the finite element solution were the exact solution, the first derivatives for each element indicated in expression 5.30 would be equal and the value of the expression would be zero. However, finite element solutions are seldom exact, so these




$$y^{(3)}(x_4) = y^{(4)}(x_4)$$

$$\left.\frac{\mathrm{d}y^{(3)}}{\mathrm{d}x}\right|_{x_4} \neq \left.\frac{\mathrm{d}y^{(4)}}{\mathrm{d}x}\right|_{x_4}$$

**Figure 5.5** Two elements joined at a node.

terms are not, in general, zero. Nevertheless, in the assembly procedure, it is assumed that, at all interior nodes, the gradient terms appear as equal and opposite from the adjacent elements and thus cancel unless an external influence acts at the node. At global boundary nodes however, the gradient terms may be specified boundary conditions or represent "reactions" obtained via the solution phase. In fact, a very powerful technique for assessing accuracy of finite element solutions is to examine the magnitude of gradient discontinuities at nodes or, more generally, interelement boundaries.

**EXAMPLE 5.5**

Use Galerkin's method to formulate a linear finite element for solving the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 4x = 0 \qquad 1 \leq x \leq 2$$

subject to $y(1) = y(2) = 0$.

■ **Solution**

First, note that the differential equation is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 4x = 0$$

which, after two direct integrations and application of boundary conditions, has the exact solution

$$y(x) = x^2 - \frac{3}{\ln 2}\ln x - 1$$

For the finite element solution, the simplest approach is to use a two-node element for which the element solution is assumed as

$$y(x) = N_1(x)y_1 + N_2(x)y_2 = \frac{x_2 - x}{x_2 - x_1}y_1 + \frac{x - x_1}{x_2 - x_1}y_2$$

where $y_1$ and $y_2$ are the nodal values. The residual equation for the element is

$$\int_{x_1}^{x_2} N_i\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 4x\right]\mathrm{d}x = 0 \qquad i = 1, 2$$

which becomes, after integration of the first term by parts,

$$N_i x \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} x \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x - \int_{x_1}^{x_2} 4xN_i\,\mathrm{d}x = 0 \qquad i = 1, 2$$

Substituting the element solution form and rearranging, we have

$$\int_{x_1}^{x_2} x \frac{\mathrm{d}N_i}{\mathrm{d}x}\left(\frac{\mathrm{d}N_1}{\mathrm{d}x}y_1 + \frac{\mathrm{d}N_2}{\mathrm{d}x}y_2\right)\mathrm{d}x = N_i x \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} 4xN_i\,\mathrm{d}x \qquad i = 1, 2$$

Expanding the two equations represented by the last result after substitution for the interpolation functions and first derivatives yields

$$\frac{1}{(x_2 - x_1)^2}\int_{x_1}^{x_2} x(y_1 - y_2)\,\mathrm{d}x = -x_1 \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x_1} - 4\int_{x_1}^{x_2} x \frac{x_2 - x}{x_2 - x_1}\,\mathrm{d}x$$

$$\frac{1}{(x_2 - x_1)^2}\int_{x_1}^{x_2} x(y_2 - y_1)\,\mathrm{d}x = x_2 \frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x_2} - 4\int_{x_1}^{x_2} x \frac{x - x_1}{x_2 - x_1}\,\mathrm{d}x$$

Integration of the terms on the left reveals the element stiffness matrix as

$$\left[k^{(e)}\right] = \frac{x_2^2 - x_1^2}{2(x_2 - x_1)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

while the gradient boundary conditions and nodal forces are evident on the right-hand side of the equations.

To illustrate, a two-element solution is formulated by taking equally spaced nodes at $x = 1,\ 1.5, 2$ as follows.

***Element 1***

$$x_1 = 1 \qquad x_2 = 1.5 \qquad k = 2.5$$

$$F_1^{(1)} = -4\int_1^{1.5} x \frac{1.5 - x}{1.5 - 1}\,\mathrm{d}x = -1.166666\ldots$$

$$F_2^{(1)} = -4\int_1^{1.5} x \frac{x - 1}{1.5 - 1}\,\mathrm{d}x = -1.33333\ldots$$

***Element 2***

$$x_1 = 1.5 \qquad x_2 = 2 \qquad k = 3.5$$

$$F_1^{(2)} = -4\int_{1.5}^{2} x \frac{2 - x}{2 - 1.5}\,\mathrm{d}x = -1.66666\ldots$$

$$F_2^{(2)} = -4\int_{1.5}^{2} x \frac{x - 1.5}{2 - 1.5}\,\mathrm{d}x = -1.83333\ldots$$

The element equations are then

$$\begin{bmatrix} 2.5 & -2.5 \\ -2.5 & 2.5 \end{bmatrix} \begin{Bmatrix} y_1^{(1)} \\ y_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} -1.1667 - \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_1} \\ -1.3333 + 1.5\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_2} \end{Bmatrix}$$

$$\begin{bmatrix} 3.5 & -3.5 \\ -3.5 & 3.5 \end{bmatrix} \begin{Bmatrix} y_1^{(2)} \\ y_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} -1.6667 - 1.5\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_2} \\ -1.8333 + 2\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_3} \end{Bmatrix}$$

Denoting the system nodal values as $Y_1, Y_2, Y_3$ at $x = 1, 1.5, 2,$ respectively, the assembled system equations are

$$\begin{bmatrix} 2.5 & -2.5 & 0 \\ -2.5 & 6 & -3.5 \\ 0 & -3.5 & 3.5 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{Bmatrix} = \begin{Bmatrix} -1.1667 - \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_1} \\ -3 \\ -1.8333 + 2\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_3} \end{Bmatrix}$$

Applying the global boundary conditions $Y_1 = Y_3 = 0,$ the second of the indicated equations gives $Y_2 = -0.5$ and substitution of this value into the other two equations yields the values of the gradients at the boundaries as

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_1} = -2.4167 \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_3} = 1.7917$$

For comparison, the exact solution gives

$$y(x = 1.5) = Y_2 = -0.5049 \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_1} = -2.3281 \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_3} = 1.8360$$

While the details will be left as an end-of-chapter problem, a four-element solution for this example (again, using equally spaced nodes $x_i \Rightarrow (1, 1.25, 1.5, 1.75, 2)$) results in the global equations

$$\begin{bmatrix} 4.5 & -4.5 & 0 & 0 & 0 \\ -4.5 & 10 & -5.5 & 0 & 0 \\ 0 & -5.5 & 12 & -6.5 & 0 \\ 0 & 0 & -6.5 & 14 & -7.5 \\ 0 & 0 & 0 & -7.5 & 7.5 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{Bmatrix} = \begin{Bmatrix} -0.5417 - \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_1} \\ -1.25 \\ -1.5 \\ -1.75 \\ -0.9583 + 2\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_5} \end{Bmatrix}$$

Applying the boundary conditions $Y_1 = Y_5 = 0$ and solving the remaining $3 \times 3$ system gives the results

$$Y_2 = -0.4026$$
$$Y_3 = -0.5047$$
$$Y_4 = -0.3603$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_1} = -2.350$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_5} = 1.831$$

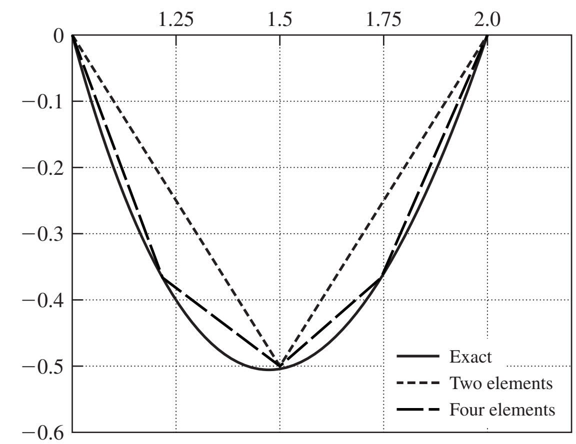



**Figure 5.6** Two-element, four-element, and exact solutions to Example 5.5.

For comparison, the exact, two-element, and four-element solutions are shown in Figure 5.6. The two-element solution is seen to be a crude approximation except at the element nodes and derivative discontinuity is significant. The four-element solution has the computed values of $y(x)$ at the nodes being nearly identical to the exact solution. With four elements, the magnitudes of the discontinuities of first derivatives at the nodes are reduced but still readily apparent.

## 5.4 APPLICATION OF GALERKIN'S METHOD TO STRUCTURAL ELEMENTS

### 5.4.1 Spar Element

Reconsidering the elastic bar or spar element of Chapter 2 and recalling that the bar is a constant strain (therefore, constant stress) element, the applicable equilibrium equation is obtained using Equations 2.29 and 2.30 as

$$\frac{\mathrm{d}\sigma_x}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(E\varepsilon_x) = E\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} = 0 \tag{5.31}$$

where we assume constant elastic modulus. Denoting element length by $L$, the displacement field is discretized by Equation 2.17:

$$u(x) = u_1N_1(x) + u_2N_2(x) = u_1\left(1 - \frac{x}{L}\right) + u_2\frac{x}{L} \tag{5.32}$$

And, since the domain of interest is the volume of the element, the Galerkin residual equations become

$$\iiint_V N_i(x)\left(E\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)\mathrm{d}V = \int_0^L N_i\left(E\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)A\,\mathrm{d}x = 0 \qquad i = 1, 2 \tag{5.33}$$

where $\mathrm{d}V = A\,\mathrm{d}x$ and $A$ is the constant cross-sectional area of the element. Integrating by parts and rearranging, we obtain

$$AE\int_0^L \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \left[N_i AE\,\frac{\mathrm{d}u}{\mathrm{d}x}\right|_0^L \tag{5.34}$$

which, utilizing Equation 5.32, becomes

$$AE\int_0^L \frac{\mathrm{d}N_1}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}x}(u_1N_1 + u_2N_2)\,\mathrm{d}x = -AE\,\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_{x=0} = -AE\,\varepsilon|_{x=0} = -A\sigma|_{x=0} \tag{5.35a}$$

$$AE\int_0^L \frac{\mathrm{d}N_2}{\mathrm{d}x}\,\frac{\mathrm{d}}{\mathrm{d}x}(u_1N_1 + u_2N_2)\,\mathrm{d}x = AE\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_{x=L} = AE\varepsilon|_{x=L} = A\sigma_{x=L} \tag{5.35b}$$

From the right sides of Equation 5.35, we observe that, for the bar element, the gradient boundary condition simply represents the applied nodal force since $\sigma A = F$.

Equation 5.35 is readily combined into matrix form as

$$AE\int_0^L \begin{bmatrix} \dfrac{\mathrm{d}N_1}{\mathrm{d}x}\dfrac{\mathrm{d}N_1}{\mathrm{d}x} & \dfrac{\mathrm{d}N_1}{\mathrm{d}x}\dfrac{\mathrm{d}N_2}{\mathrm{d}x} \\ \dfrac{\mathrm{d}N_1}{\mathrm{d}x}\dfrac{\mathrm{d}N_2}{\mathrm{d}x} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x}\dfrac{\mathrm{d}N_2}{\mathrm{d}x} \end{bmatrix}\mathrm{d}x\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{5.36}$$

where the individual terms of the matrix are integrated independently.

Carrying out the indicated differentiations and integrations, we obtain

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{5.37}$$

which is the same result as obtained in Chapter 2 for the bar element. This simply illustrates the equivalence of Galerkin's method and the methods of equilibrium and energy (Castigliano) used earlier for the bar element.

### 5.4.2 Beam Element

Application of the Galerkin method to the beam element begins with consideration of the equilibrium conditions of a differential section taken along the

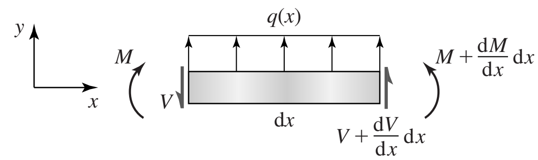



**Figure 5.7** Differential section of a loaded beam.

longitudinal axis of a loaded beam as depicted in Figure 5.7 where $q(x)$ represents a distributed load expressed as force per unit length. Whereas $q$ may vary arbitrarily, it is assumed to be constant over a differential length $\mathrm{d}x$. The condition of force equilibrium in the $y$ direction is

$$-V + \left(V + \frac{\mathrm{d}V}{\mathrm{d}x}\,\mathrm{d}x\right) + q(x)\,\mathrm{d}x = 0 \tag{5.38}$$

from which

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q(x) \tag{5.39}$$

Moment equilibrium about a point on the left face is expressed as

$$M + \frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x - M + \left(V + \frac{\mathrm{d}V}{\mathrm{d}x}\,\mathrm{d}x\right)\mathrm{d}x + [q(x)\,\mathrm{d}x]\,\frac{\mathrm{d}x}{2} = 0 \tag{5.40}$$

which (neglecting second-order differentials) gives

$$\frac{\mathrm{d}M}{\mathrm{d}x} = -V \tag{5.41}$$

Combining Equations 5.39 and 5.41, we obtain

$$\frac{\mathrm{d}^2M}{\mathrm{d}x^2} = q(x) \tag{5.42}$$

Recalling, from the elementary strength of materials theory, the *flexure formula* corresponding to the sign conventions of Figure 5.7 is

$$M = EI_z\frac{\mathrm{d}^2v}{\mathrm{d}x^2} \tag{5.43}$$

(where in keeping with the notation of Chapter 4, $v$ represents displacement in the $y$ direction), which in combination with Equation 5.42 provides the governing equation for beam flexure as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_z\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\right) = q(x) \tag{5.44}$$

Galerkin's finite element method is applied by taking the displacement solution in the form

$$v(x) = N_1(x)v_1 + N_2(x)\theta_1 + N_3(x)v_2 + N_4(x)\theta_2 = \sum_{i=1}^{4} N_i(x)\delta_i \qquad (5.45)$$

as in Chapter 4, using the interpolation functions of Equation 4.26. Therefore, the element residual equations are

$$\int_{x_1}^{x_2} N_i(x)\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_z\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right) - q(x)\,\mathrm{d}x\right] = 0 \qquad i = 1, 4 \qquad (5.46)$$

Integrating the derivative term by parts and assuming a constant $EI_z$, we obtain

$$N_i(x)EI_z\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}\bigg|_{x_1}^{x_2} - EI_z\int_{x_1}^{x_2}\frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}\,\mathrm{d}x - \int_{x_1}^{x_2} N_i q(x)\,\mathrm{d}x = 0 \qquad i = 1, 4 \qquad (5.47)$$

and since

$$V = -\frac{\mathrm{d}M}{\mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(EI_z\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right) = -EI_z\frac{\mathrm{d}^3 v}{\mathrm{d}x^3} \qquad (5.48)$$

we observe that the first term of Equation 5.47 represents the shear force conditions at the element nodes. Integrating again by parts and rearranging gives

$$EI_z\int_{x_1}^{x_2}\frac{\mathrm{d}^2 N_i}{\mathrm{d}x^2}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,\mathrm{d}x = \int_{x_1}^{x_2} N_i q(x)\,\mathrm{d}x - N_i EI_z\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}\bigg|_{x_1}^{x_2} + \frac{\mathrm{d}N_i}{\mathrm{d}x}EI_z\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\bigg|_{x_1}^{x_2} \qquad i = 1, 4 \qquad (5.49)$$

and, per Equation 5.43, the last term on the right introduces the moment conditions at the element boundaries. Integration by parts was performed twice in the preceding development for reasons similar to those mentioned in the context of the bar element. By so doing, the order of the two derivative terms appearing in the first integral in Equation 5.49 are the same, and the resulting stiffness matrix is thus symmetric, and the shear forces and bending moments at element nodes now explicitly appear in the element equations.

Equation 5.49 can be written in the matrix form $[k]\{\delta\} = \{F\}$ where the terms of the stiffness matrix are defined by

$$k_{ij} = EI_z\int_{x_1}^{x_2}\frac{\mathrm{d}^2 N_i}{\mathrm{d}x^2}\frac{\mathrm{d}^2 N_j}{\mathrm{d}x^2}\,\mathrm{d}x \qquad i, j = 1, 4 \qquad (5.50)$$

which is identical to results previously obtained by other methods. The terms of the element force vector are defined by

$$F_i = \int_{x_1}^{x_2} N_i q(x)\,\mathrm{d}x - N_i EI_z \frac{\mathrm{d}^3 v}{\mathrm{d}x^3}\bigg|_{x_1}^{x_2} + \frac{\mathrm{d}N_i}{\mathrm{d}x} EI_z \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\bigg|_{x_1}^{x_2} \qquad i = 1, 4 \tag{5.51a}$$

or, using Equations 5.43 and 5.48,

$$F_i = \int_{x_1}^{x_2} N_i q(x)\,\mathrm{d}x + N_i V(x)|_{x_1}^{x_2} + \frac{\mathrm{d}N_i}{\mathrm{d}x} M(x)|_{x_1}^{x_2} \qquad i = 1, 4 \tag{5.51b}$$

where the integral term represents the equivalent nodal forces and moments produced by the distributed load. If $q(x) = q =$ constant (positive upward), substitution of the interpolation functions into Equation 5.51 gives the element nodal force vector as

$$\{F\} = \begin{Bmatrix} \dfrac{qL}{2} - V_1 \\ \dfrac{qL^2}{12} - M_1 \\ \dfrac{qL}{2} + V_2 \\ -\dfrac{qL^2}{12} + M_2 \end{Bmatrix} \tag{5.52}$$

Where two beam elements share a common node, one of two possibilities occurs regarding the shear and moment conditions:

1. If no external force or moment is applied at the node, the shear and moment values of Equation 5.52 for the adjacent elements are equal and opposite, cancelling in the assembly step.
2. If a concentrated force is applied at the node, the sum of the boundary shear forces for the adjacent elements must equal the applied force.

Similarly, if a concentrated moment is applied, the sum of the boundary bending moments must equal the applied moment. Equation 5.52 shows that the effects of a distributed load are allocated to the element nodes. Finite element software packages most often allow the user to specify a "pressure" on the transverse face of the beam. The specified pressure actually represents a distributed load and is converted to the nodal equivalent loads in the software.

## 5.5 ONE-DIMENSIONAL HEAT CONDUCTION

Application of the Galerkin finite method to the problem of one-dimensional, steady-state heat conduction is developed with reference to Figure 5.8a, which depicts a solid body undergoing heat conduction in the direction of the $x$ axis

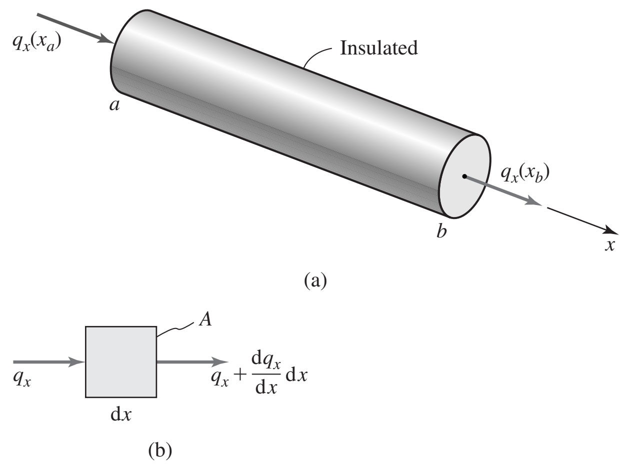



**Figure 5.8** Insulated body in one-dimensional heat conduction.

only. Surfaces of the body normal to the $x$ axis are assumed to be perfectly insulated, so that no heat loss occurs through these surfaces. Figure 5.8b shows the control volume of differential length d$x$ of the body, which is assumed to be of constant cross-sectional area and uniform material properties. The principle of conservation of energy is applied to obtain the governing equation as follows:

$$E_{\text{in}} + E_{\text{generated}} = E_{\text{increase}} + E_{\text{out}} \tag{5.53}$$

Equation 5.53 states that the energy entering the control volume plus energy generated internally by any heat source present must equal the increase in internal energy plus the energy leaving the control volume. For the volume of Figure 5.8b, during a time interval d$t$, Equation 5.53 is expressed as

$$q_x A\,\mathrm{d}t + QA\,\mathrm{d}x\,\mathrm{d}t = \Delta U + \left(q_x + \frac{\partial q_x}{\partial x}\mathrm{d}x\right) A\,\mathrm{d}t \tag{5.54}$$

where

$q_x$ = heat flux across boundary (W/m$^2$, Btu/hr-ft$^2$);
$Q$ = internal heat generation rate (W/m$^3$, Btu/hr-ft$^3$);
$U$ = internal energy (W, Btu).

The last term on the right side of Equation 5.54 is a two-term Taylor series expansion of $q_x(x, t)$ evaluated at $x + \mathrm{d}x$. Note the use of partial differentiation, since for now, we assume that the dependent variables vary with time as well as spatial position.

The heat flux is expressed in terms of the temperature gradient via Fourier's law of heat conduction:

$$q_x = -k_x \frac{\partial T}{\partial x} \tag{5.55}$$

where $k_x$ = material thermal conductivity in the $x$ direction (W/m-°C, Btu/hr-ft-°F) and $T = T(x, t)$ is temperature. The increase in internal energy is

$$\Delta U = c\rho A \, \mathrm{d}x \, \mathrm{d}T \tag{5.56}$$

where

$c$ = material specific heat (J/kg-°C, Btu/slug-°F);
$\rho$ = material density (kg/m$^3$, slug/ft$^3$).

Substituting Equations 5.55 and 5.56 into 5.54 gives

$$QA \, \mathrm{d}x \, \mathrm{d}t = c\rho A \, \mathrm{d}x \, \mathrm{d}T + \frac{\partial}{\partial x}\left(-k_x \frac{\partial T}{\partial x}\right) A \, \mathrm{d}x \, \mathrm{d}t \tag{5.57}$$

Assuming that the thermal conductivity is constant, Equation 5.57 becomes

$$Q = c\rho \frac{\partial T}{\partial t} - k_x \frac{\partial T^2}{\partial x^2} \tag{5.58}$$

For now we are interested only in steady-state heat conduction and for the steady state $\partial T/\partial t = 0$, so the governing equation for steady-state, one-dimensional conduction is obtained as

$$k_x \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} + Q = 0 \tag{5.59}$$

Next, the Galerkin finite element method is applied to Equation 5.59 to obtain the element equations. A two-node element with linear interpolation functions is used and the temperature distribution in an element expressed as

$$T(x) = N_1(x)T_1 + N_2(x)T_2 \tag{5.60}$$

where $T_1$ and $T_2$ are the temperatures at nodes 1 and 2, which define the element, and the interpolation functions $N_1$ and $N_2$ are given by Equation 5.20. As in previous examples, substitution of the discretized solution (5.60) into the governing differential Equation 5.55 results in the residual integrals:

$$\int_{x_1}^{x_2} \left(k_x \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} + Q\right) N_i(x) A \, \mathrm{d}x = 0 \qquad i = 1, 2 \tag{5.61}$$

where we note that the integration is over the volume of the element, that is, the domain of the problem, with $\mathrm{d}V = A \, \mathrm{d}x$.

Integrating the first term by parts (for reasons already discussed) yields

$$k_x A N_i(x) \frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x_1}^{x_1} - k_x A \int_{x_1}^{x_2} \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}T}{\mathrm{d}x} \, \mathrm{d}x + A \int_{x_1}^{x_2} Q N_i(x) \, \mathrm{d}x = 0 \qquad i = 1, 2 \tag{5.62}$$

Evaluating the first term at the limits as indicated, substituting Equation 5.60 into the second term, and rearranging, Equation 5.58 results in the two equations

$$k_x A \int_{x_1}^{x_2} \frac{\mathrm{d}N_1}{\mathrm{d}x}\left(\frac{\mathrm{d}N_1}{\mathrm{d}x}T_1 + \frac{\mathrm{d}N_2}{\mathrm{d}x}T_2\right)\mathrm{d}x = A\int_{x_1}^{x_2} QN_1\,\mathrm{d}x - k_x A\left.\frac{\mathrm{d}T}{dx}\right|_{x_1} \tag{5.63}$$

$$k_x A \int_{x_1}^{x_2} \frac{\mathrm{d}N_2}{\mathrm{d}x}\left(\frac{\mathrm{d}N_1}{\mathrm{d}x}T_1 + \frac{\mathrm{d}N_2}{\mathrm{d}x}T_2\right)\mathrm{d}x = A\int_{x_1}^{x_2} QN_2\,\mathrm{d}x + k_x A\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x_2} \tag{5.64}$$

Equations 5.63 and 5.64 are of the form

$$[k]\{T\} = \{f_Q\} + \{f_g\} \tag{5.65}$$

where $[k]$ is the element conductance ("stiffness") matrix having terms defined by

$$k_{lm} = k_x A \int_{x_1}^{x_2} \frac{\mathrm{d}N_l}{\mathrm{d}x}\frac{\mathrm{d}N_m}{\mathrm{d}x}\,\mathrm{d}x \qquad l, m = 1, 2 \tag{5.66}$$

The first term on the right-hand side of Equation 5.65 is the nodal "force" vector arising from internal heat generation with values defined by

$$\begin{aligned} f_{Q1} &= A\int_{x_1}^{x_2} QN_1\,\mathrm{d}x \\ f_{Q2} &= A\int_{x_1}^{x_2} QN_2\,\mathrm{d}x \end{aligned} \tag{5.67}$$

and vector $\{f_g\}$ represents the gradient boundary conditions at the element nodes. Performing the integrations indicated in Equation 5.66 gives the conductance matrix as

$$[k] = \frac{k_x A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{5.68}$$

while for constant internal heat generation $Q$, Equation 5.67 results in the nodal vector

$$\{f_Q\} = \begin{Bmatrix} \dfrac{QAL}{2} \\ \dfrac{QAL}{2} \end{Bmatrix} \tag{5.69}$$

The element gradient boundary conditions, using Equation 5.55, described by

$$\{f_g\} = k_x A \begin{Bmatrix} \left.\dfrac{-\mathrm{d}T}{\mathrm{d}x}\right|_{x_1} \\ \left.\dfrac{\mathrm{d}T}{\mathrm{d}x}\right|_{x_2} \end{Bmatrix} = A \begin{Bmatrix} q|_{x_1} \\ -q|_{x_2} \end{Bmatrix} \tag{5.70}$$

are such that, at internal nodes where elements are joined, the values for the adjacent elements are equal and opposite, cancelling mathematically. At external nodes, that is, at the ends of the body being analyzed, the gradient values may be specified as known heat flux input and output or computed if the specified boundary condition is a temperature. In the latter case, the gradient computation is analogous to computing reaction forces in a structural model. Also note that the area is a common term in the preceding equations and, since it is assumed to be constant over the element length, could be ignored in each term. However, as will be seen in later chapters when we account for other heat transfer conditions, the area should remain in the equations as defined. These concepts are illustrated in the following example.

## EXAMPLE 5.6

The circular rod depicted in Figure 5.9 has an outside diameter of 60 mm, length of 1 m, and is perfectly insulated on its circumference. The left half of the cylinder is aluminum, for which $k_x = 200$ W/m-°C and the right half is copper having $k_x = 389$ W/m-°C. The extreme right end of the cylinder is maintained at a temperature of 80°C, while the left end is subjected to a heat input rate 4000 W/m$^2$. Using four equal-length elements, determine the steady-state temperature distribution in the cylinder.

### ■ Solution

The elements and nodes are chosen as shown in the bottom of Figure 5.9. For aluminum elements 1 and 2, the conductance matrices are

$$[k_{\mathrm{al}}] = \frac{k_x A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{200(\pi/4)(0.06)^2}{0.25}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2.26\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{W/°C}$$

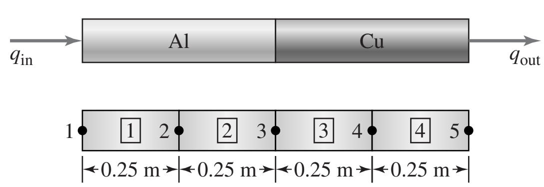



**Figure 5.9** Circular rod of Example 5.6.

while, for copper elements 3 and 4,

$$[k_{\text{cu}}] = \frac{k_x A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{389(\pi/4)(0.06)^2}{0.25}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4.40\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{W/}^\circ\text{C}$$

Applying the end conditions $T_5 = 80°\text{C}$ and $q_1 = 4000\ \text{W/m}^2$, the assembled system equations are

$$\begin{bmatrix} 2.26 & -2.26 & 0 & 0 & 0 \\ -2.26 & 4.52 & -2.26 & 0 & 0 \\ 0 & -2.26 & 6.66 & -4.40 & 0 \\ 0 & 0 & -4.40 & 8.80 & -4.40 \\ 0 & 0 & 0 & -4.40 & 4.40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ 80 \end{Bmatrix} = \begin{Bmatrix} 4000 \\ 0 \\ 0 \\ 0 \\ -q_5 \end{Bmatrix} \frac{\pi(0.06)^2}{4}$$

$$= \begin{Bmatrix} 11.31 \\ 0 \\ 0 \\ 0 \\ -0.0028q_5 \end{Bmatrix}$$

Accounting for the known temperature at node 5, the first four equations can be written as

$$\begin{bmatrix} 2.26 & -2.26 & 0 & 0 \\ -2.26 & 4.52 & -2.26 & 0 \\ 0 & -2.26 & 6.66 & -4.40 \\ 0 & 0 & -4.40 & 8.80 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 11.31 \\ 0 \\ 0 \\ 352.0 \end{Bmatrix}$$

The system of equations is triangularized (used here simply to illustrate another solution method) by the following steps. Replace the second equation by the sum of the first and second to obtain

$$\begin{bmatrix} 2.26 & -2.26 & 0 & 0 \\ 0 & 2.26 & -2.26 & 0 \\ 0 & -2.26 & 6.66 & -4.40 \\ 0 & 0 & -4.40 & 8.80 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 11.31 \\ 11.31 \\ 0 \\ 352.0 \end{Bmatrix}$$

Next, replace the third equation by the sum of the second and third

$$\begin{bmatrix} 2.26 & -2.26 & 0 & 0 \\ 0 & 2.26 & -2.26 & 0 \\ 0 & 0 & 4.40 & -4.40 \\ 0 & 0 & -4.40 & 8.80 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 11.31 \\ 11.31 \\ 11.31 \\ 352.0 \end{Bmatrix}$$

Finally, replace the fourth with the sum of the third and fourth to obtain

$$\begin{bmatrix} 2.26 & -2.26 & 0 & 0 \\ 0 & 2.26 & -2.26 & 0 \\ 0 & 0 & 4.40 & -4.40 \\ 0 & 0 & 0 & 4.40 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 11.31 \\ 11.31 \\ 11.31 \\ 363.31 \end{Bmatrix}$$

The triangularized system then gives the nodal temperatures in succession as

$$T_4 = 82.57^\circ\text{C}$$

$$T_3 = 85.15^\circ\text{C}$$

$$T_2 = 90.14^\circ\text{C}$$

$$T_1 = 95.15^\circ\text{C}$$

The fifth equation of the system is

$$-4.40T_4 + 4.40(80) = -0.0028q_5$$

which, on substitution of the computed value of $T_4$, results in

$$q_5 = 4038.6 \text{ W/m}^2$$

As this is assumed to be a steady-state situation, the heat flow from the right-hand end of the cylinder, node 5, should be exactly equal to the inflow at the left end. The discrepancy in this case is due simply to round-off error in the computations, which were accomplished via a hand calculator for this example. If the values are computed to "machine accuracy" and no intermediate rounding is used, the value of the heat flow at node 5 is found to be exactly 4000 W/m$^2$. In fact, it can be shown that, for this example, the finite element solution is exact.

## 5.6 CLOSING REMARKS

The method of weighted residuals, especially the embodiment of the Galerkin finite element method, is a powerful mathematical tool that provides a technique for formulating a finite element solution approach to practically any problem for which the governing differential equation and boundary conditions can be written. For situations in which a principle such as the first theorem of Castigliano or the principle of minimum potential energy is applicable, the Galerkin method produces exactly the same formulation. In subsequent chapters, the Galerkin method is extended to two- and three-dimensional cases of structural analysis, heat transfer, and fluid flow. Prior to examining specific applications, we examine, in the next chapter, the general requirements of interpolation functions for the formulation of a finite element approach to any type of problem.

## REFERENCES

1. Stasa, F. L. *Applied Element Analysis for Engineers.* New York: Holt, Rinehart, and Winston, 1985.
2. Burnett, D. S. *Finite Element Analysis.* Reading, MA: Addison-Wesley, 1987.

# PROBLEMS

**5.1** Verify the integration and subsequent determination of $c_1$ in Example 5.1.

**5.2** Using the procedure discussed in Example 5.4, determine three trial functions for the problem of Example 5.1.

**5.3** It has been stated that the trial functions used in the method of weighted residuals generally satisfy the physics of the problem described by the differential equation to be solved. Why does the trial function assumed in Example 5.3 not satisfy the physics of the problem?

**5.4** For each of the following differential equations and stated boundary conditions, obtain a one-term solution using Galerkin's method of weighted residuals and the specified trial function. In each case, compare the one-term solution to the exact solution.

a.
$$\frac{d^2y}{dx^2} + y = 2x \qquad 0 \leq x \leq 1$$
$$y(0) = 0$$
$$y(1) = 0$$
$$N_1(x) = x(1 - x^2)$$

b.
$$\frac{d^2y}{dx^2} + y = 2\sin x \qquad 0 \leq x \leq 1$$
$$y(0) = 0$$
$$y(1) = 0$$
$$N_1(x) = \sin \pi x$$

c.
$$\frac{dy}{dx} + y^2 = 4x \qquad 0 \leq x \leq 1$$
$$y(0) = 0$$
$$y(1) = 0$$
$$N_1(x) = x^2(1 - x)$$

d.
$$\frac{dy}{dx} - y = 2 \qquad 0 \leq x \leq 10$$
$$y(0) = 0$$
$$y(10) = 0$$
$$N_1(x) = x^2(10 - x)^2$$

e.
$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = x \qquad 0 \leq x \leq 1$$
$$y(0) = 0$$
$$y(1) = 0$$
$$N_1(x) = x(x - 1)^2$$

**5.5(a)–(e)** For each of the differential equations given in Problem 5.4, use the method of Example 5.4 to determine the trial functions for a two-term approximate solution using Galerkin's method of weighted residuals.

**5.6** For the four-element solution of Example 5.5, verify the correctness of the assembled system equations. Apply the boundary conditions and solve the reduced system of equations. Compute the first derivatives at each node of each element. Are the derivatives continuous across the nodal connections between elements?

**5.7** Each of the following differential equations represents a physical problem, as indicated. For each case given, formulate the finite element equations (that is, determine the stiffness matrix and load vectors) using Galerkin's finite element method for a two-node element of length $L$ with the interpolation functions

$$N_1(x) = 1 - \frac{x}{L} \qquad N_2(x) = \frac{x}{L}$$

a. One-dimensional heat conduction with linearly varying internal heat generation

$$k_x A \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} + Q_0 A x = 0$$

where $k_x$, $Q_0$, and $A$ are constants.

b. One-dimensional heat conduction with surface convection

$$k_x A \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} - hPT = hPT_a$$

where $k_x$, $Q_0$, $A$, $h$, $P$, and $T_a$ are constants.

c. Torsion of an elastic circular cylinder

$$JG \frac{\mathrm{d}^2 \theta}{\mathrm{d}x^2} = 0$$

where $J$ and $G$ are constants.

**5.8** A two-dimensional beam is subjected to a linearly varying distributed load given by $q(x) = q_0 x$, $0 \leq x \leq L$, where $L$ is total beam length and $q_0$ is a constant. For a finite element located between nodes at arbitrary positions $x_i$ and $x_j$ so that $x_i < x_j$ and $L_e = x_j - x_i$ is the element length, determine the components of the force vector at the element nodes using Galerkin's finite element method. (Note that this is simply the last term in Equation 5.47 adjusted appropriately for element location.)

**5.9** Repeat Problem 5.8 for a quadratrically distributed load $q(x) = q_0 x^2$.

**5.10** Considering the results of either Problem 5.8 or 5.9, are the distributed loads allocated to element nodes on the basis of static equilibrium? If your answer is no, why not and how is the distribution made?

**5.11** A tapered cylinder that is perfectly insulated on its periphery is held at constant temperature 212°F at $x = 0$ and at temperature 80°F at $x = 4$ in. The cylinder diameter varies from 2 in. at $x = 0$ to 1 in. at $x = L = 4$ in. per Figure P5.11. The conductance coefficient is $k_x = 64$ Btu/hr – ft – °F. Formulate a four-element finite element model of this problem and solve for the nodal temperatures and

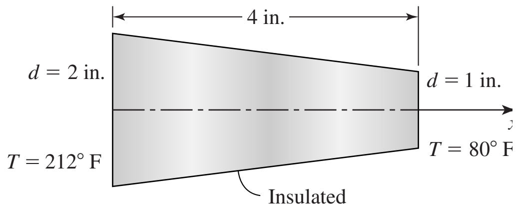



**Figure P5.11**

the heat flux values at the element boundaries. Use Galerkin's finite element method.

**5.12** Consider a tapered uniaxial tension-compression member subjected to an axial load as shown in Figure P5.12. The cross-sectional area varies as $A = A_0(1 - x/2L)$, where $L$ is the length of the member and $A_0$ is the area at $x = 0$. Given the governing equation

$$E\frac{d^2u}{dx^2} = 0$$

as in Equation 5.31, obtain the Galerkin finite element equations per Equation 5.33.

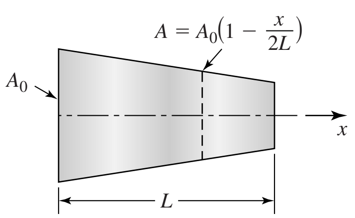



**Figure P5.12**

**5.13** Many finite element software systems have provision for a tapered beam element. Beginning with Equation 5.46, while noting that $I_z$ is not constant, develop the finite element equations for a tapered beam element.

**5.14** Use the results of Problem 5.13 to determine the stiffness matrix for the tapered beam element shown in Figure P5.14.

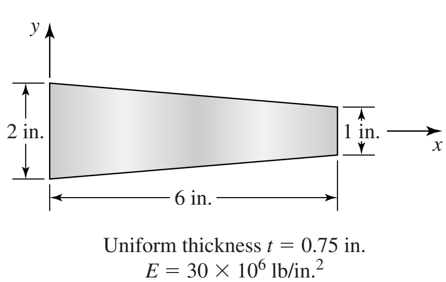



**Figure P5.14**

**5.15** Consider a two-dimensional problem governed by the differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

(this is Laplace's equation) in a specified two-dimensional domain with specified boundary conditions. How would you apply the Galerkin finite element method to this problem?

**5.16** Reconsider Equation 5.24. If we do *not* integrate by parts and simply substitute the discretized solution form, what is the result? Explain.

**5.17** Given the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4y = x$$

Assume the solution as a power series

$$y(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots$$

and obtain the relations governing the coefficients of the power series solution. How does this procedure compare to the Galerkin method?

**5.18** The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 3 \qquad 0 \le x \le 1$$

has the exact solution

$$y(x) = 3 + Ce^{-x}$$

where $C$ = constant. Assume that the domain is $0 \le x \le 1$ and the specified boundary condition is $y(0) = 0$. Show that, if the procedure of Example 5.4 is followed, the exact solution is obtained.